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Topological Charges of Noncommutative Soliton

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Abstract

The noncommutative soliton is characterized by the use of the projection operators in non-commutative space which is closely related to the K-theory of C^*-algebra. We consider the variations of projection operators in the moduli space along the commutative directions and discuss their topological charges. When applied to the string theory, it gives the modification of the brane charges embedded in the world volume due to tachyon background.

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1 Introduction

In the recent development of string theory, the study of the D-brane has been one of the most fruitful source of the inspirations. One of the interesting issues among them is to understand the dynamical process such as brane-anti-brane pair annihilation. Research toward such direction was pioneered by A. Sen [1] and the rôle of tachyon field was clarified. Inspired by these works, E. Witten [2] proposed that the D-brane charge should be measured by the K-group instead of the cohomology. It is based on the fact that D-brane carries the information of the vector bundle due to the massless gauge fields on it.

A conceptual progress was made by the recent discovery of the non-commutative soliton [3]. The solution is characterized by the use of the projection operators which reflects the very nature of the noncommutativity. It was applied immediately [4][5] to the string theory as a mechanism to understand the tachyon condensation. The simplification in the large non-commutative limit helps the analysis by justifying to neglect many stringy corrections in the lagrangian.

In mathematical viewpoint, the use of projection operators are quite suggestive since they play a fundamental rôle to study the geometry of the non-commutative space. In the context of the K-theory of C^*-algebra (see for example [6]), the K_0 group of a C^*-algebra $\mathcal{A}$ is identified as the equivalence class of the formal differences of the projection operators in $\mathcal{A}$. In this viewpoint, the use of projection operators in the tachyon condensation gives a natural passage from topological to C^*-algebraic description of K-theory.

In [7] and the references therein, we can find the topological invariants constructed in arbitrary C^* algebra. The simplest example is the rank $(= Tr(\Pi))$ of the projection operator Π. In the context of the tachyon condensation [4], it was identified as the number of the D-branes of the lower dimensions which are created by the tachyon condensation process (we call it "the descendent D-brane" in the following). More general invariants can be constructed by pairing the projection operators with the elements of the cyclic cohomology of the C^*-algebra $\mathcal{A}$,

$$\int \mathrm{Tr}\left(\Pi(d\Pi)^{2n}\right) \ , \tag{1}$$

for positive integers n. Here d is the "derivative" operator which defines the cyclic cohomology. Conne used this pairing to define the non-commutative version of the index theorem.

One purpose of this note is to identify the D-brane interpretation of the topological charges (1). In the tachyon condensation process, the relevant C^*-algebra is identified as $\mathcal{A} \equiv C^\infty(M) \otimes \mathcal{B}(\mathcal{H})$ where M is the world volume for the descendent brane and $\mathcal{B}(\mathcal{H})$ is the linear operators acting on the Hilbert space $\mathcal{H}$ of the harmonic oscillators. It is actually the simplest example as the cohomology of C^*-algebra. In fact, it is well-known [6] that the $K_0(\mathcal{A})$ is isomorphic to $K^0(M)$, the *topological* K group of the manifold M. In other word, there is a direct correspondence between the projection operator in $\mathcal{A}$ and the vector bundle on M. We identify the projection operator as defining noncommutative soliton and the vector bundle as those of the descendent brane. In this context, the derivative d in (1) reduces to the ordinary exterior derivative along M and the charges (1) becomes the characteristic class of the vector bundle on M. This correspondence gives additional insights, namely the rôle of the tachyon background, to the origin of the gauge symmetry on the descendent D-brane.

2 Noncommutative soliton and topology

To describe the idea in more physical language, we start from the scalar field theory in $(2q+2)$ spatial dimensions ($q \geq 1$) slightly extending [3] by introducing extra commutative directions. We assume there are two non-commutative directions (described by coordinates $y, \bar{y}$) and $2q$ commutative directions described by x^a.

For the static configuration, the energy functional is given as,

$$E = \frac{1}{g^2} \int d^{2q}x d^2y \, (\frac{1}{2}\partial_x\phi\partial_x\phi + \frac{1}{2}\partial_y\phi\partial_{\bar{y}}\phi + V(*\phi)) \ . \tag{2}$$

Here $*$-product is defined by a non-commutativity parameter θ as

$$A * B = e^{\frac{\theta}{2}(\partial_y\partial_{\bar{y}'} - \partial_{y'}\partial_{\bar{y}})} A(x:y,\bar{y}) B(x:y',\bar{y}')\Big|_{y=y',\bar{y}=\bar{y}'} \tag{3}$$

In large θ limit, we rescale, $y \to \sqrt{\theta}y$, $\bar{y} \to \sqrt{\theta}\bar{y}$. After the rescale, the first and the third terms in (2) are multiplied by θ and in the infinite θ limit the second term can be neglected.

GMS soliton is constructed by the field configuration satisfying

$$\phi_0(y) * \phi_0(y) = \phi_0(y) \ . \tag{4}$$

Indeed if λ_* gives the minimum of the potential $\frac{\partial V(\lambda)}{\partial \lambda}\Big|_{\lambda=\lambda_*} = 0$, $\phi(y) = \lambda_* \phi_0(y)$ also minimizes the potential, as long as $V(\lambda)$ is a polynomial of λ.

For the explicit construction of the configuration which satisfies (4), GMS used one-to-one correspondence between the space of the functions of the non-commutative coordinates (say $\mathcal{F}$) and the space of the linear operators $\mathcal{B}(\mathcal{H})$ acting on the Hilbert space $\mathcal{H}$ of the harmonic oscillators. The correspondence uses the Weyl ordering and is defined as,

$$\begin{aligned} f(y,\bar{y}) \in \mathcal{F} \quad &\leftrightarrow \quad \mathcal{O}_f \in \mathcal{B}(\mathcal{H}) \\ \mathcal{O}_f \quad &= \quad \frac{1}{2\pi}\int d^2k \tilde{f}(k,\bar{k}) e^{2\pi i(k\bar{a}^\dagger + \bar{k}a)} \\ \tilde{f}(k,\bar{k}) \quad &\equiv \quad \int d^2y f(y,\bar{y}) e^{-2\pi i(y\bar{k}+k\bar{y})} \ . \end{aligned} \tag{5}$$

The Moyal product in $\mathcal{F}$ is translated into the ordinary product in $\mathcal{B}(\mathcal{H})$,

$$\mathcal{O}_f \cdot \mathcal{O}_g = \mathcal{O}_{f*g} \ . \tag{6}$$

Therefore in $\mathcal{B}(\mathcal{H})$, (4) is translated into the conventional condition of the projections, $\Pi^2 = \Pi$ for $\Pi = \mathcal{O}_{\phi_0}$.

In $\mathcal{B}(\mathcal{H})$ the construction of the projections is straightforward since we know the orthonormal basis $|k\rangle = \frac{{a^\dagger}^k}{\sqrt{k!}}|0\rangle$. GMS introduced the rank k solution as the wave function corresponding to,

$$\Pi_k \equiv \sum_{r=0}^{k-1} |r\rangle\langle r| \ . \tag{7}$$

By putting it into the potential, the energy can be evaluated as

$$\int d^2y V(\lambda_* \phi_k(y,\bar{y})) = \mathrm{Tr}_{\mathcal{H}}(V(\lambda_*)\Pi_k) = V(\lambda_*)k \ , \tag{8}$$

where we denote $\phi_k(y,\bar{y})$ as the wave function corresponding to Π_k.

Actually there are infinite number of projection operators which have the same potential energy. They are obtained by twisting projection operators by $g \in U(\mathcal{H})$ where $U(\mathcal{H})$ is the set of unitary operators acting on $\mathcal{H}$,[1]

$$\Pi_k^g = g \Pi_k g^{-1} \ . \tag{9}$$

[1] GMS has shown that the kinetic term favors the choice $g = 1$ in finite θ. In our context, it will help to give a unique solution to each topologically disconnected sectors of projection operators.

The set of rank k projection is thus parametrized by the infinite dimensional grassmannian

$$Gr_k(\mathcal{H}) \equiv \{\text{set of } k \text{ dimensional subspaces in } \mathcal{H}\} \ , \tag{10}$$

since Π_k^g is specified by the k-dimensional subspace to which projection is defined. These spaces have rich topology. For example when $k = 1$ it is simplified to the infinite dimensional projective space $CP(\mathcal{H})$ which has the homotopy group, $\pi_{2q}(CP(\mathcal{H})) = \mathbf{Z}$ for $q = 0, 1, 2, 3 \cdots$.

In the presence of the extra commuting space M, one may consider the variations of the projection operators along that direction. This is, of course, valid only if the potential is dominant enough compared to the energy coming from the variation in x directions. For that purpose, one may rewrite $V(\phi)$ as $tV(\phi)$ and takes $t \to \infty$ limit. In this limit, one may restrict the configuration space of ϕ to the space of rank k projections in $\mathcal{H}$ and the configuration space of the scalar field theory in $(2q + 2) + 1$ dimensions is reduced to the non-linear sigma model in $2q + 1$ dimensions whose target space is $Gr_k(\mathcal{H})$. This is physically interpreted as the Nambu-Goldstone bosons associated with the symmetry breaking $U(\infty) \to U(k) \times U(\infty - k)$.

From the mathematical viewpoint, they give the general idempotent elements in $C^\infty(M) \otimes \mathcal{B}(\mathcal{H})$ and should be included to the analysis of the K-theory of C^*-algebra. As we wrote in the introduction, there is a correspondence between the projection operator of $C^\infty(M) \otimes \mathcal{B}(\mathcal{H})$ and the vector bundle over M. From the projection operator Π, we define the fiber bundle on M by assigning fiber $\Pi_x(\mathcal{H})$ for each point $x \in M$.

Actually $Gr_k(\mathcal{H})$ is the *classifying space* of rank k vector bundles. Namely any vector bundle over M is isomorphic to the vector bundle thus defined by the projection operator. The homotopy class of the vector bundle is classified by the mapping

$$M \to Gr_k(\mathcal{H}) \ . \tag{11}$$

In this sense, there is a one-to-one correspondence between the homotopy class of the vector bundle over M and the connected components of the configuration space of the non-linear sigma model.

The remaining task is to identify (1) with the characteristic class of the vector bundle by using the projection operators. This is an elementary material but let us write down explicitly. We need to introduce define the covariant derivative. by using the projections.

Since the fiber bundle $\mathcal{H} \to M$ itself is trivial and just the direct product, all the topological non-trivialities come from the projection onto the finite dimensional subspace. The covariant derivative in such a situation is defined as,

$$D \equiv \Pi(x) \cdot d \cdot \Pi(x). \tag{12}$$

To relate it to the conventional definition of the covariant derivative, we introduce the coordinate dependent orthonormal basis,

$$\langle i|j\rangle = \delta_{ij}, \quad \Pi(x) = \sum_{i=0}^{k-1} |i\rangle\langle i| \ . \tag{13}$$

The sections of the vector bundle is written locally as $f(x) = \sum_{i=0}^{k-1} f_i(x)|i\rangle$. The covariant derivative acts on it as

$$Df(x) = \sum_{i=0}^{k-1} (Df)_i |i\rangle \ , \quad Df_i = df_i + \sum_{j=0}^{k-1} A_{ij} f_j \quad A_{ij} = \langle i|d|j\rangle \ . \tag{14}$$

A_{ij} gives the $U(k)$ gauge connection which is an analogue of the non-abelian Berry phase [8]. The local change of the basis $|i\rangle$ gives the gauge transformation. This gauge symmetry is originated from the configuration space of projection operators namely the Grassmaniann $Gr_k(\mathcal{H})$. It is well-known that there is a "hidden" gauge symmetry of gauge group H for the coset space nonlinear sigma model on G/H.

The curvature two form associated with this covariant derivative is,

$$F = D^2 = \Pi d\Pi d\Pi \ . \tag{15}$$

From this expression, it is straightforward to write the characteristic class associated with the vector bundle defined by the projection. For example, p-th Chern class is given by (1), $c_p \propto \int \mathrm{Tr}\, F^p = \int \mathrm{Tr}\, \Pi\, (d\Pi)^{2p}$.

3 Application to string theory

In the string theory, the commuting directions are identified with the world volume of the descendent D-brane. Let us first consider the open bosonic string which has boundaries at one D-$(p+2)$ brane [4]. In this context, the scalar field is replaced by the tachyon field $T(x, y)$ and there is also the noncommutative $U(1)$ gauge field $A_\mu(x, y)$ on the brane. The action is

$$S = \int dtd^pxd^2y \left(f(*T)(DT)^2 - V(*T) + g(*T)(F_{\mu\nu})^2 + \cdots \right). \qquad (16)$$

The functions $f(T), V(T), g(T)$ can be in principle determined by the string field theory [11]. As conjectured by Sen [10], the information to construct non-commutative soliton is supplied by the "universality" of these functions. It is conjectured that there are two critical points $T = 0, t_*$ in $V(T)$. When $T = t_*$, $V(t_*)$ is equal to the tension of the D-branes. This point is the ordinary perturbative vacuum of the open string. When $T = 0$, we have $f(0) = g(0) = V(0) = 0$. This is the point where the D-brane is annihilated to vacuum and there are no propagating open string degree of freedom. For the recent development the "nothing state", see [12].

The noncommutative soliton solution for the tachyon field is given by,

$$T(x, y) = t_* \cdot \phi(y) + 0 \cdot (1 - \phi(y)). \qquad (17)$$

Here $\phi(y)$ is the configuration corresponding to degree k projection Π_k. Following to Witten's notation [9], we denote V as the subspace in $\mathcal{H}$ which is defined by the projection Π_k and W is specified by $1 - \Pi_k$. In [4], it is shown that the soliton solution can be naturally identified as k D-p branes which emerge after the tachyon condensation.

Around this vacuum, the fluctuation of tachyon fields and the gauge fields can be expanded in terms of $\phi_{nm}(y)$ which corresponds to $|n\rangle\langle m|$ in $\mathcal{B}(\mathcal{H})$. In particular, $U(1)$-connection A_μ in $p + 2$-brane can be identified as $U(\mathcal{H})$-connection on p-brane. By sandwiching it by the projection operators Π and $1 - \Pi$, we get four sectors. Namely for $\mathcal{O} \in \mathcal{B}(\mathcal{H})$, we write

$$\begin{aligned} \mathcal{O}_{VV} &\equiv \Pi\mathcal{O}\Pi \ , & \mathcal{O}_{VW} &\equiv \Pi\mathcal{O}(1 - \Pi) \ , \\ \mathcal{O}_{WV} &\equiv (1 - \Pi)\mathcal{O}\Pi \ , & \mathcal{O}_{WW} &\equiv (1 - \Pi)\mathcal{O}(1 - \Pi) \ . \end{aligned} \qquad (18)$$

Components in VW, WV, and WW sectors become non-propagating after the tachyon condensation since they are connected with the nothing states. The VV sector, on the other hand, describes the surviving physical modes in the descendent D-brane.

At this point we introduce the position dependent projection operators. As in the field theory example, it induces the gauge symmetry on the world volume of the descendent brane. Since we already have such gauge symmetry from VV part of the $U(\mathcal{H})$ gauge field A [4], the position dependent projection operators introduce the modifications of the covariant derivative,

$$\Pi(d+A)\Pi \equiv \Pi D_A \Pi = \Pi \, d\, \Pi + A_{VV} \ , \quad A_{VV} \equiv \Pi A \Pi. \tag{19}$$

From this expression, the effective curvature is obtained as,

$$\mathcal{F} = \Pi (D_A \Pi)^2 = F_{VV} + \Pi (d\Pi)^2 + \cdots \ . \tag{20}$$

Here $F_{VV} = dA_{VV} + A_{VV} \wedge A_{VV}$ is the curvature form from A_{VV}. The second term is the correction of the curvature from the tachyon configuration.

We may understand the implication of this formula in the following way. We have two equivalent routes to realize a nontrivial vector bundle over the descendent D-brane created by the tachyon condensation. One track is to start from the nontrivial $U(\mathcal{H})$ bundle and apply the constant projection. The other is to apply the twisted projection to the trivial $A = 0$ background. The first viewpoint has a merit to understand the relation between gauge fields in the original D-brane and the descendant. On the other hand the second approach is better to understand the relation with the K-theory of the C^*-algebra.

At this point, it is pedagogical to indicate the analogue in the 't Hooft-Polyakov monopoles [13] where Higgs field varies along the spatial infinity and so is the projection to the unbroken $U(1)$ part. For the calculation of the monopole charge, one needs to modify the $U(1)$ field strength by the Higgs field. For example, in the simplest $SU(2) \to U(1)$ case, the effective field strength is given as,

$$\mathcal{F}_{\mu\nu} = F_{n\mu\nu} \hat{\phi}_n - \epsilon_{nml} \hat{\phi}_n D_\mu \hat{\phi}_m D_\nu \hat{\phi}_l \ . \tag{21}$$

Here the Higgs field ($\hat{\phi}_n \equiv \phi_n / \sqrt{\phi^2}$) takes their value in S^2 where the Higgs potential is minimized. The inclusion of the latter term is essential to evaluate the monopole charge as the winding number of $\pi_2(S^2)$.

The correspondence between K-theory of C^*-algebra and tachyon condensation becomes more accurate if we consider the pair annihilation of D-$\bar{D}$ system in the superstring theory. In commutative situation, we already know [2] that the formal difference between the vector bundles defined on D and $\bar{D}$ branes defines the topological K-group, i. e. the D-brane charges in M. In the noncommutative case, similar definition of K-group is possible as the formal difference between the two projection operators.

Such an idea was illustrated in the pair annihilation of $D\bar{D}$ system in [9]. We start from a pair of D-$(p+2)$ brane and $\bar{D}$-$(p+2)$ brane and introduce the large non-commutativity in two directions. In this case, we have two copies of the same Hilbert space of the harmonic oscillators on D- (resp$\bar{D}$-) brane. We denote them as $\mathcal{H}$ and $\bar{\mathcal{H}}$. The tachyon fields $\sigma, \bar{\sigma}$ appear as the *interpolating* linear map between the two Hilbert spaces,

$$\sigma : \mathcal{H} \to \bar{\mathcal{H}} \qquad \bar{\sigma} : \bar{\mathcal{H}} \to \mathcal{H} \ , \tag{22}$$

with the finite dimensional kernel and cokernel (i.e. Fredholm operators). We note that the tachyon fields themselves can not be idempotent as in the bosonic string. Witten [9] has shown that they should instead satisfy

$$\sigma\bar{\sigma}\sigma = \sigma \qquad \bar{\sigma}\sigma\bar{\sigma} = \bar{\sigma} \ , \tag{23}$$

to recover the equation of motion of the string field theory. From such operator, one may construct two projection operators acting on $\mathcal{H}$ and $\bar{\mathcal{H}}$,

$$\Pi = 1 - \bar{\sigma}\sigma \ , \quad \bar{\Pi} = 1 - \sigma\bar{\sigma} \ . \tag{24}$$

It is easy to observe that Π (resp. $\bar{\Pi}$) defines the projection to the kernel V (resp. cokernel W) of σ in $\mathcal{H}$ ($\bar{\mathcal{H}}$)

$$\Pi^2 = \Pi, \quad \bar{\Pi}^2 = \bar{\Pi}, \quad \Pi\bar{\sigma} = \bar{\Pi}\sigma = 0 \ . \tag{25}$$

The dimensions of V and W are identified as the number of descendant D-p ($\bar{D}$-p) branes after the tachyon condensation. The index of σ,

$$\mathrm{Ind}(\sigma) = \dim(\mathrm{Ker}\ \sigma) - \dim(\mathrm{Coker}\ \sigma) = \mathrm{Tr}_{\mathcal{H}}\,\Pi - \mathrm{Tr}_{\bar{\mathcal{H}}}\,\bar{\Pi} \ . \tag{26}$$

is the total D-p brane number.

We may repeat our generalization where the partial isometry σ varies along the world volume. As in the case of the bosonic string, Π (resp. $\bar{\Pi}$) defines a vector bundle on p-brane world volume M. The left hand side of (26) becomes the index bundle, namely the formal difference between two vector bundles $[\mathrm{Ker}(\sigma)] - [\mathrm{Coker}(\sigma)]$ over M. Their isomorphic class precisely define an element of (topological) $K^0(M)$.

In the presence of the non-constant tachyon field σ, the D-brane charges embedded in the descendant brane should be also modified. The result is completely parallel to the bosonic case. The field strengths on D and $\bar{D}$ branes should be modified to

$$\mathcal{F} = \Pi(D_A\Pi)^2 \ , \quad \bar{\mathcal{F}} = \bar{\Pi}(D_{\bar{A}}\bar{\Pi})^2 \ . \tag{27}$$

Such a change forces us to modify the D-brane charges on the world volume. They are evaluated from Chern-Simons coupling[14] with modified gauge potential,

$$\int_M C \wedge (\mathrm{Tr}_{\mathcal{H}}(e^{\mathcal{F}}) - \mathrm{Tr}_{\bar{\mathcal{H}}}(e^{\bar{\mathcal{F}}})). \tag{28}$$

We note that it will be useful if this formula can be "derived" from the generic result for the commutative case derived by [15] who used the concept of the superconnection [16],

$$\int C \wedge \mathrm{STr} e^{\mathcal{F}} \ , \quad \mathcal{F} = \begin{pmatrix} F_{VV} - T\bar{T} & DT \\ \overline{DT} & F_{WW} - \bar{T}T \end{pmatrix} \ . \tag{29}$$

4 Future directions

There are a few directions which we would like to proceed in the future. One direction is the generalization of the C^*-algebra and the cyclic cohomology. Originally Conne introduced such machinery to study the geometrical objects which is not accessible from the topological methods. There are a lot of examples which seem to be described in the context of noncommutative solitons. The second direction is the use of $K_1(\mathcal{A})$ group. In the context of C^* algebra, they are defined as the unitary operators on the algebra. In the commutative context, Horava [17] argued that tachyon field defines the unitary transformation that defines the vector bundle on the brane. It would be critically important to find the analogue of noncommutative solitons constructed out of the unitaries. For the current understanding to this direction, we can find many interesting suggestions in [18].

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